



teletransmissionsteori
Tekniska Högskolan i Lund, 220 07 Lund

ON ESTIMATION OF THE AMPLITUDE
AND THE PHASE FUNCTION

Per Eriksson
—

Technical Report TR-148
March 1981
—



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41553424_0055

ON ESTIMATION OF THE AMPLITUDE AND THE PHASE FUNCTION

Per Eriksson



Author:
Title:
Keywords:

Subject:
Date:

Language:
Country:
City:

Document number:
Filing number:
Engineering number:
Date: 1-24-67 11:30 AM



Per Eriksson

Dokumenttitel och undertitel

On Estimation of the Amplitude and the Phase function

Referat (sammmandrag)

ABSTRACT

In this paper we will study the problem of estimating the amplitude and the phase function of a linear system disturbed by additive noise. We use the sine signal as an input. An analog maximum likelihood estimate of the amplitude and the phase is found. We also find a maximum likelihood estimate for the discrete time situation. This estimation procedure can be made extremely simple if the ratio between the sine signal frequency and the sampling rate is $1/4$. Finally we note that one may consider the estimation procedure as a bandpass filter, the bandwidth of which becomes smaller as the estimation time increases.

Referat skrivet av

Author

Förslag till ytterligare nyckelord

Klassifikationssystem och -klass(er)

Indextermer (ange källa)

Omfång

15 pages

Språk

English

Sekretessuppgifter

Övriga bibliografiska uppgifter

ISSN

ISBN

Dokumentet kan erhållas från

Telecommunication Theory, Electrical
Engineering Department, University of
Lund, S-220 07 LUND 7 / SWEDEN

Mottagarens uppgifter

Pris

CONTENTS

ABSTRACT

1. INTRODUCTION	1
2. THE ESTIMATION PROBLEM	2
3. THE ANALOG ML ESTIMATES	3
4. THE DISCRETE TIME ML ESTIMATES	8
5. REFERENCES	15

ABSTRACT

In this paper we will study the problem of estimating the amplitude and the phase function of a linear system disturbed by additive noise. We use the sine signal as an input. An analog maximum likelihood estimate of the amplitude and the phase is found. We also find a maximum likelihood estimate for the discrete time situation. This estimation procedure can be made extremely simple if the ratio between the sine signal frequency and the sampling rate is $1/4$. Finally we note that one may consider the estimation procedure as a bandpass filter, the bandwidth of which becomes smaller as the estimation time increases.

1. INTRODUCTION

This is the second report of a project on acoustical channels supported by the National Swedish Environment Protection Board.

In this paper we will study certain characteristics of communication channels. The channels here are mathematical models of physical channels. Typically, they could be telephone lines, radio links, or acoustical channels.

We restrict our studies to channels that are to be described by a stable linear time-invariant system followed by an addition of noise. This is illustrated in Fig. 1.1.

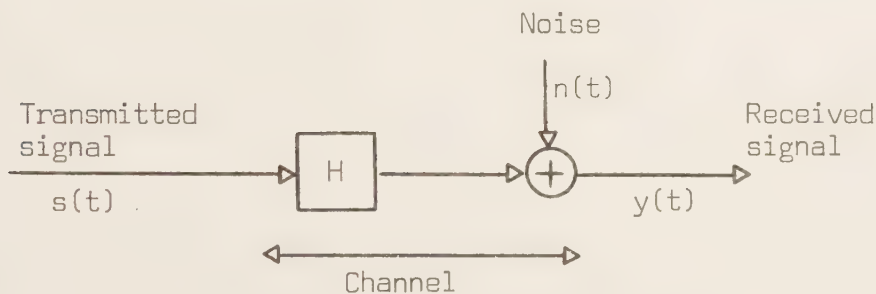


Figure 1.1 The mathematical model of the channels.

The system H is completely described by its impulse response $h(t)$ or the Fourier transform of $h(t)$ denoted by $H(f)$. The noise signal $n(t)$ is assumed to be a zero-mean, wide-sense stationary, Gaussian random process with the spectral density given by

$$R_N(f) = \frac{N_0}{2} \quad (1.1)$$

2. THE ESTIMATION PROBLEM

We now want to estimate the amplitude function $A(f)$ defined by

$$A(f) = |H(f)| \quad (2.1)$$

and the phase function $\varphi(f)$ defined by

$$\varphi(f) = \arg\{H(f)\}. \quad (2.2)$$

This gives the relation

$$H(f) = A(f) e^{j\varphi(f)}. \quad (2.3)$$

We will investigate some methods to estimate the amplitude function and the phase function with one frequency at a time. Here we therefore use the sine signal as transmitted signal. That is

$$s(t) = \sin 2\pi f_0 t \quad \text{for } t \in \{-\infty, \infty\} \quad (2.4)$$

where the frequency f_0 is a fixed and known parameter.

The received signal $y(t)$ is given by

$$y(t) = A(f_0) \sin(2\pi f_0 t + \varphi(f_0)) + n(t). \quad (2.5)$$

This signal is observed in the time interval $0 \leq t \leq T_0$. The problem is therefore to estimate $A(f_0)$ and $\varphi(f_0)$ by means of the received signal $y(t)$ in the time interval $0 \leq t \leq T_0$. We assume $A(f_0)$ and $\varphi(f_0)$ to be nonrandom parameters and look for maximum likelihood (ML) estimates $\hat{A}_{ML}(f_0)$ and $\hat{\varphi}_{ML}(f_0)$.

3. THE ANALOG ML ESTIMATES

To find the ML estimates we must find the absolute maximum of the log likelihood function. If we omit the factor $\frac{1}{N_0}$ this function is given by [1]

$$D(A_0, \varphi_0) = 2 \int_0^{T_0} y(t) A_0 \sin(2\pi f_0 t + \varphi_0) dt - \int_0^{T_0} [A_0 \sin(2\pi f_0 t + \varphi_0)]^2 dt \quad (3.1)$$

where $A_0 = A(f_0)$ and $\varphi_0 = \varphi(f_0)$.

Since the frequency f_0 is a known parameter, we may choose the time T_0 to be an integer number K of half-periods i.e.

$$T_0 = \frac{K}{2f_0} \quad (3.2)$$

The function $D(A_0, \varphi_0)$ is then given by

$$D(A_0, \varphi_0) = 2A_0 \int_0^{T_0} y(t) \sin(2\pi f_0 t + \varphi_0) dt - \frac{1}{2} A_0^2 T_0. \quad (3.3)$$

We simplify the notation by using

$$S = \frac{2}{T_0} \int_0^{T_0} y(t) \sin(2\pi f_0 t) dt \quad (3.4)$$

and

$$C = \frac{2}{T_0} \int_0^{T_0} y(t) \cos(2\pi f_0 t) dt. \quad (3.5)$$

This gives

$$D(A_0, \varphi_0) = A_0 T_0 (S \cos \varphi_0 + C \sin \varphi_0) - \frac{1}{2} A_0^2 T_0. \quad (3.6)$$

Necessary, but not sufficient, conditions for the absolute maximum of $D(A_0, \varphi_0)$ are

$$\frac{\partial D(A_0, \varphi_0)}{\partial A_0} = 0 \quad (3.7)$$

and

$$\frac{\partial D(A_0, \varphi_0)}{\partial \varphi_0} = 0. \quad (3.8)$$

This gives

$$A_0 = (S \cos \varphi_0 + C \sin \varphi_0) \quad (3.9)$$

and

$$S \sin \varphi_0 = C \cos \varphi_0. \quad (3.10)$$

Since in these points

$$\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0^2} = -T_0 < 0 \quad (3.11)$$

and

$$\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0^2} \cdot \frac{\partial^2 D(A_0, \varphi_0)}{\partial \varphi_0^2} - \left(\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0 \partial \varphi_0} \right)^2 = (A_0 T_0)^2 > 0 \quad (3.12)$$

we have found the maximum points of $D(A_0, \varphi_0)$ given by the solutions to (3.9) and (3.10). Unfortunately (3.9) and (3.10) neither guarantee positive amplitude A_0 nor a phase φ_0 in the entire interval $0 \leq \varphi_0 < 2\pi$. But this is easily corrected and with this restriction we find the absolute maximum of $D(A_0, \varphi_0)$ in the point

$$\hat{A}_{ML}(f_0) = \sqrt{S^2 + C^2} \quad (3.13)$$

and

$$\hat{\varphi}_{ML}(f_0) = \arg\{S + jC\}. \quad (3.14)$$

The estimation procedure can be implemented as shown in Fig. 3.1.

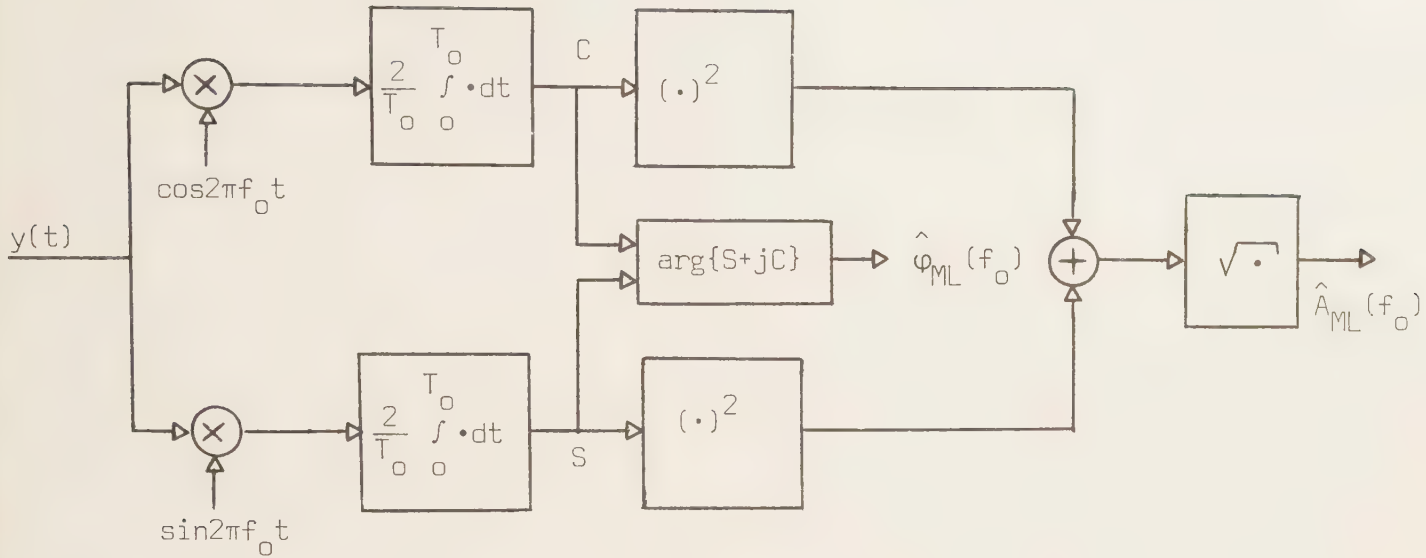


Figure 3.1. The estimation procedure.

By means of today's electrical integrated analog circuits the estimation procedure may be realized with accuracy in practise.

It is well-known [1] that if one only wants to estimate $A(f_0)$, (3.13) will still be the ML estimate provided that $\varphi(f_0)$ is a random variable with the probability density function

$$p(\varphi) = \frac{1}{2\pi} \quad \text{for } -\pi \leq \varphi < \pi \quad (3.15)$$

In this paper we have besides (3.13) also got an ML estimate of the phase and we don't have to make the assumption (3.15).

Bias and variance of the estimate

In order to find out how effective this estimation procedure is we will compute the bias and the variance of the estimate. We restrict our study to the amplitude estimate.

Let us rewrite (3.13) as

$$\begin{aligned}
 \hat{A}_{ML}(f_0) &= \sqrt{(A_0 \cos \varphi_0 + \xi_1)^2 + (A_0 \sin \varphi_0 + \xi_2)^2} = \\
 &= A_0 \sqrt{1 + \frac{2\xi}{A_0} + \frac{\xi_1^2 + \xi_2^2}{A_0^2}}
 \end{aligned} \tag{3.16}$$

where

$$\xi_1 = \frac{2}{T_0} \int_0^{T_0} n(t) \sin(2\pi f_0 t) dt, \tag{3.17}$$

$$\xi_2 = \frac{2}{T_0} \int_0^{T_0} n(t) \cos(2\pi f_0 t) dt \tag{3.18}$$

and

$$\xi = \xi_1 \cos \varphi_0 + \xi_2 \sin \varphi_0. \tag{3.19}$$

The assumptions made for the noise signal $n(t)$ in chapter 1 give us the following relations:

$$E\{\xi_1\} = E\{\xi_2\} = E\{\xi\} = 0, \tag{3.20}$$

$$\sigma_\xi^2 = E\{\xi_1^2\} = E\{\xi_2^2\} = E\{\xi^2\} = \frac{N_0}{T_0} \tag{3.21}$$

and

$$E\{\xi_1 \xi_2\} = 0 \tag{3.22}$$

Furthermore we note that ξ_1 , ξ_2 and ξ are Gaussian random variables. We now assume that T_0 is sufficiently large to make

$$\frac{\sigma_\xi^2}{A_0^2} = \frac{N_0}{T_0 A_0^2} \ll 1 \tag{3.23}$$

valid.

Then we may use the approximation

$$\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2} \epsilon - \frac{1}{8} \epsilon^2 \quad (3.24)$$

in (3.16). This gives

$$\begin{aligned} E\{\hat{A}_{ML}(f_o)\} &\simeq A_o E\left\{1 + \frac{\xi}{A_o} + \frac{\xi_1^2 + \xi_2^2}{2A_o^2} - \frac{1}{8} \left[\frac{4\xi^2}{A_o^2} + \frac{\xi_1^4 + \xi_2^4 + 2\xi_1^2 \xi_2^2}{A_o^4} + \frac{4\xi(\xi_1^2 + \xi_2^2)}{A_o^3} \right] \right\} = \\ &= A_o \left[1 + \frac{\sigma_\xi^2}{A_o^2} - \frac{1}{8} \left(\frac{4\sigma_\xi^2}{A_o^2} + \frac{8\sigma_\xi^4}{A_o^4} \right) \right] \simeq A_o \left[1 + \frac{\sigma_\xi^2}{2A_o^2} \right] \end{aligned} \quad (3.25)$$

where σ_ξ^2 is defined by (3.21).

This means that the estimate is biased. The bias is

$$b \simeq \frac{\sigma_\xi^2}{2A_o} \quad (3.26)$$

The variance of the estimate is given by

$$\begin{aligned} \text{Var}\{\hat{A}_{ML}(f_o)\} &= E\{(\hat{A}_{ML}(f_o))^2\} - (E\{\hat{A}_{ML}(f_o)\})^2 \simeq \\ &\simeq A_o^2 + 2\sigma_\xi^2 - A_o^2 \left(1 + \frac{\sigma_\xi^2}{A_o^2} + \frac{\sigma_\xi^4}{4A_o^4} \right) \simeq \sigma_\xi^2 = \frac{N_o}{T_o} \end{aligned} \quad (3.27)$$

The relative bias is

$$b_R = \frac{b}{A_o} = \frac{\sigma_\xi^2}{2A_o^2} = \frac{N_o}{2T_o A_o^2} \quad (3.28)$$

and the relative variance is

$$\text{Var}_R\{\hat{A}_{ML}(f_o)\} = \frac{\text{Var}\{\hat{A}_{ML}(f_o)\}}{A_o^2} \simeq \frac{\sigma_\xi^2}{A_o^2} = \frac{N_o}{T_o A_o^2} \quad (3.29)$$

4. THE DISCRETE TIME ML ESTIMATES

We look for a discrete time estimation procedure and therefore we have to sample the received signal. In order to avoid aliasing errors we insert an ideal lowpass filter with the cutoff frequency f_i and choose the sampling rate f_s to be

$$f_s = 2f_i \quad (4.1)$$

The cutoff frequency must be chosen so that

$$f_o < f_i. \quad (4.2)$$

The situation is illustrated in Fig. 4.1.

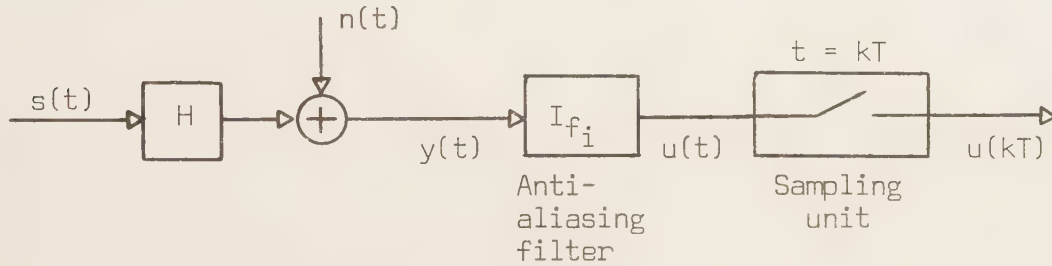


Figure 4.1. The discrete time approach.

In Fig. 4.1 k is an integer number and

$$f_s = 1/T. \quad (4.3)$$

The signal after the anti-aliasing filter I_{f_i} is denoted $u(t)$. This gives

$$u(kT) = A_o \sin(2\pi f_o kT + \varphi_o) + n_u(kT) \quad (4.4)$$

where $n_u(kT)$ is a discrete time random process.

We are now able to formulate the discrete time estimation problem. The observations of $u(kT)$ are made for $k = 1, 2, \dots, L$ and the task is to find the ML estimates of $A(f_o)$ and $\varphi(f_o)$.

Since the assumption made for the noise signal $n(t)$ in chapter 1 is still valid $n_u(kT)$ becomes a zero-mean, wide sense stationary, Gaussian random

process. Furthermore we have

$$E\{n_u(kT) n_u(kT+\ell T)\} = \begin{cases} \sigma_n^2 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \neq 0 \end{cases} \quad (4.5)$$

where ℓ is an integer number and

$$\sigma_n^2 = N_o f_s = \frac{N_o f_s}{2} \quad (4.6)$$

To find the ML estimates we must find the absolute maximum of the log likelihood function given by

$$D(A_o, \varphi_o) = 2 \sum_{k=1}^L u(kT) A_o \sin(2\pi f_o kT + \varphi_o) - \sum_{k=1}^L A_o^2 \sin^2(2\pi f_o kT + \varphi_o). \quad (4.7)$$

Throughout we will use the corresponding old notation from chapter 3 with renewed meaning. Let us define S and C by

$$S = \frac{2}{L} \sum_{k=1}^L u(kT) \sin(2\pi f_o kT) \quad (4.8)$$

and

$$C = \frac{2}{L} \sum_{k=1}^L u(kT) \cos(2\pi f_o kT) \quad (4.9)$$

We may now choose L and f_s so that the sums are made over an integer number of half-periods, i.e.

$$\frac{L}{f_s} = \frac{M}{2f_o} \quad (4.10)$$

where M is an integer number.

Necessary, but not sufficient, conditions for the maximum of $D(A_o, \varphi_o)$ are

$$\frac{\partial D(A_o, \varphi_o)}{\partial A_o} = 0 \quad (4.11)$$

and

$$\frac{\partial D(A_0, \varphi_0)}{\partial \varphi_0} = 0. \quad (4.12)$$

This gives

$$A_0 = (S \cos \varphi_0 + C \sin \varphi_0) \quad (4.13)$$

and

$$S \sin \varphi_0 = C \cos \varphi_0 \quad (4.14)$$

Since

$$\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0^2} = -L < 0 \quad (4.15)$$

and

$$\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0^2} \cdot \frac{\partial^2 D(A_0, \varphi_0)}{\partial \varphi_0^2} - \left(\frac{\partial^2 D(A_0, \varphi_0)}{\partial A_0 \partial \varphi_0} \right)^2 = (A_0 L)^2 > 0 \quad (4.16)$$

we have found the maximum points of $D(A_0, \varphi_0)$ given by the solutions to (4.13) and (4.14). In order to guarantee a positive amplitude estimate and a phase estimate in the entire interval $0 \leq \hat{\varphi}_{ML}(f_0) < 2\pi$, the ML estimates are given by

$$\hat{A}_{ML}(f_0) = \sqrt{S^2 + C^2} \quad (4.17)$$

and

$$\hat{\varphi}_{ML}(f_0) = \arg\{S + jC\}. \quad (4.18)$$

This discrete time estimation procedure is shown in Fig. 4.1.

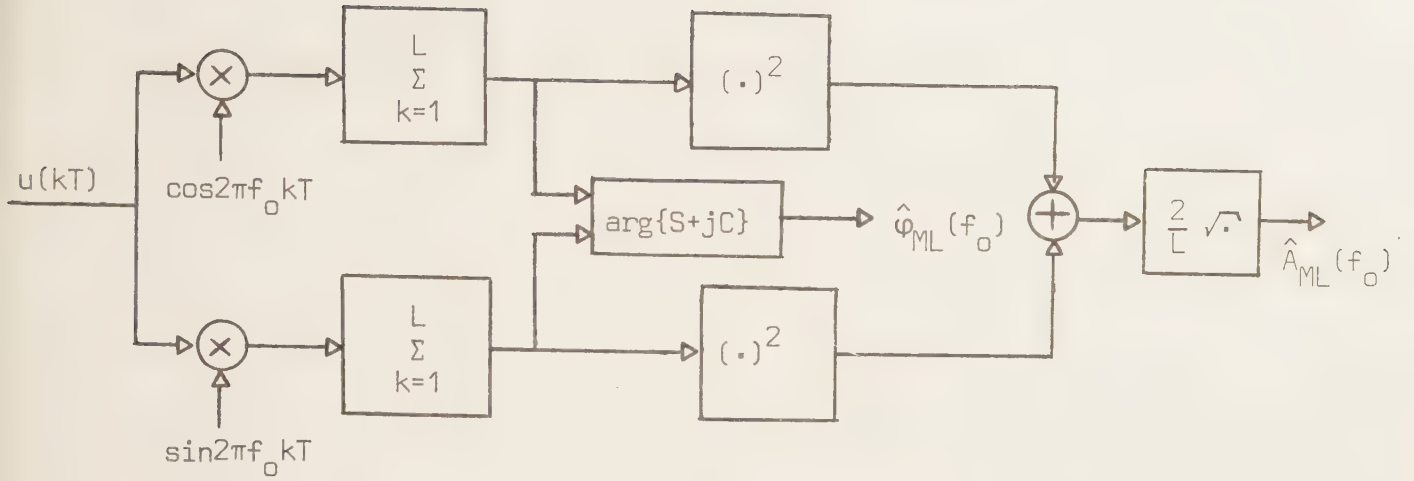


Figure 4.1. The discrete time estimation procedure.

Bias and variance of the estimate

In order to find out how effective this discrete time estimation procedure is, we will compute the bias and the variance of the estimate. We restrict our study to the amplitude estimate. To be able to compare the analog with the discrete time estimation procedure we will prescribe observation over the same number of half-periods i.e.

$$T_o = LT. \quad (4.19)$$

Let us rewrite (4.1) as

$$\begin{aligned} \hat{A}_{ML}(f_o) &= \sqrt{(A_o \cos \varphi_o + \xi_1)^2 + (A_o \sin \varphi_o + \xi_2)^2} = \\ &= A_o \sqrt{1 + \frac{\xi_1^2 + \xi_2^2}{A_o^2} + \frac{2\xi}{A_o}} \end{aligned} \quad (4.20)$$

where ξ_1 , ξ_2 and ξ are defined as

$$\xi_1 = \frac{2}{L} \sum_{k=1}^L n_u(kT) \sin 2\pi f_o kT, \quad (4.21)$$

$$\xi_2 = \frac{2}{L} \sum_{k=1}^L n_u(kT) \cos 2\pi f_o kT \quad (4.22)$$

and

$$\xi = \xi_1 \sin \varphi_0 + \xi_2 \cos \varphi_0 \quad (4.23)$$

The assumptions made for the noise signal $n(t)$ in chapter 1 give us the following relations:

$$E\{\xi_1\} = E\{\xi_2\} = E\{\xi\} = 0 \quad (4.24)$$

$$\sigma_\xi^2 = E\{\xi_1^2\} = E\{\xi_2^2\} = E\{\xi^2\} = \frac{2\sigma_n^2}{L} = \frac{2N_0 f_i}{L} = \frac{N_0 f_s}{L} \quad (4.25)$$

and

$$E\{\xi_1 \xi_2\} = 0 \quad (4.26)$$

Fürthermore we note that ξ_1 , ξ_2 and ξ are Gaussian random variables.

We now make use of (4.19) which gives

$$\sigma_\xi^2 = \frac{N_0 f_s}{L} = \frac{N_0}{LT} = \frac{N_0}{T_0} \quad (4.27)$$

This means that the analog and the discrete time estimate have exactly the same statistical property.

We now observe that it is very convenient if

$$f_s = 4f_0 \quad (4.28)$$

since then

$$\cos(2\pi f_0 kT) = \cos(k \frac{\pi}{2}) \quad (4.29)$$

and

$$\sin(2\pi f_0 kT) = \sin(k \frac{\pi}{2}). \quad (4.30)$$

The relation (4.28) is valid with extremely high accuracy if f_s and f_o are generated from the same stable clock as shown in Fig. 4.2.



Figure 4.2. Synchronization of f_s and f_o .

After the divider we place a filter that gives us $\sin 2\pi f_o t$.

The estimation procedure is simplified. This is shown in Fig. 4.3.

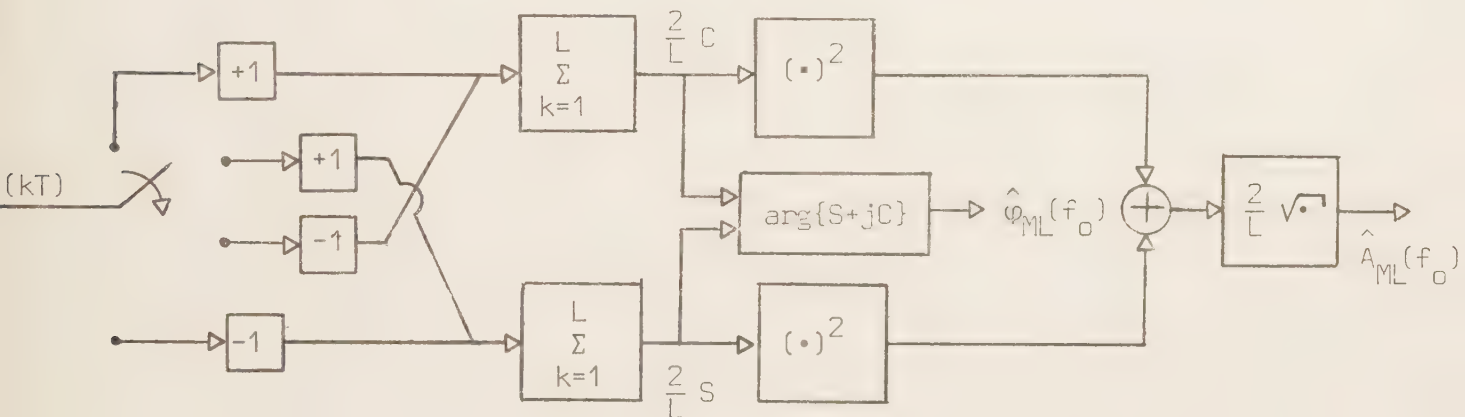


Figure 4.3. Simplified estimation procedure.

This estimation procedure is very simple and easy to implement. We note that the estimation procedure can be made with very high accuracy in practice.

The estimation procedure may be considered as a band-pass filter with a center frequency f_o and a bandwidth proportional to $1/LT$ [2]. If the noise is not white, not Gaussian or the anti-aliasing filter not ideal, the procedure still works even if the estimates do not remain ML estimates. The aim of the anti-aliasing filter is essentially to suppress noise and this means that the design of the filter is not critical. If the input signal is not a pure sine but a period signal, f_o , with harmonics, we note that the even harmonics are cancelled by the estimation procedure if $L/4$ is an

integer number. The odd harmonics have to be suppressed by the anti-aliasing filter. Finally we note that in practice it could be convenient to estimate amplitude and phase even of the input signal. It is due to the fact that this might be the simplest and most accurate way to control the input signal.

5. REFERENCES

- [1] Harry L. van Trees
Detection, Estimation, and Modulation Theory.
Part I.
Wiley, New York, 1968.

- [2] K.J. Åström
Lectures on System Identification, Chapter 3.
Report 7504, Department of Automatic Control,
Lund Institute of Technology.

LIST OF REPORTS FROM TTT

TR-131	Aulin Tor	Symbol Error Probability Bounds for Coherently Viterbi Detected Digital FM.
TR-132	Sörnmo Leif Börjesson Per Ola Nygårds Mats-Erik Pahlm Olle	A method for evaluating the properties of QRS shape features.
TR-133	Eriksson Per	On Estimation of Power Attenuation of a Noisy Channel.
TR-134	Eriksson Per	On Estimation of the Amplitude and the Phase Function.
TR-135	Mandersson Bengt	On the properties of the convergence of adaptive soft least squares filters.
TR-136	Börjesson Per Ola	Non-linear prediction for data compression of quantized signals with application to Gaussian processes and to ECGs.
TR-137	Börjesson Per Ola Pahlm Olle Sörnmo Leif Nygårds Mats-Erik	An adaptive QRS detector based on maximum-a-posteriori estimation.
TR-138	Aulin Tor Sundberg Carl-Erik	Differential Detection of Continuous Phase modulated signals.
TR-139	Aulin Tor Sundberg Carl-Erik	Further Results on M-ary Multi-h Constant Envelope Digital FM Signals.
TR-140	Aulin Tor Lindell Göran Sundberg Carl-Erik	Partial Response CPM Schemes with Short Pulses - Tradeoff Between Error Probability and Spectrum.
TR-141	Aulin Tor Sundberg Carl-Erik	Digital FM Spectra - Numerical Calculations and Asymptotic Behaviour.
TR-142	Anderson J.B. Sundberg Carl-Erik Aulin Tor Rydbeck Nils	Power-Bandwidth performance of smoothed phase modulation codes.
TR-143	Eriksson Per Salomonsson Göran Claesson Ingvar Lind Ingmar	On the Design of Adaptive Receivers Using a Frequency Domain Approach.
TR-144	Aulin Tor Sundberg Carl-Erik Svensson Arne	Simple Viterbi detectors for partial response continuous phase modulated signals.
TR-145	Sörnmo Leif	RESERAPPORT. - Computers in Cardiology, Williamsburg, Virginia, 22-24 October 1980. - En studieresa i USA.
TR-146	Sundberg Carl-Erik	Autocorrelation and crosscorrelation properties of sequences generated from cyclic error-correcting codes.
TR-147	Aulin Tor Sundberg Carl-Erik	RESERAPPORT. - National Telecommunications Conference, Houston, Texas, 30 nov - 4 dec 1980, - Int. Symposium on Information Theory, Santa Monica, California, 9-12 febr. 1981, - Studieresor i USA och Canada dec 1980 och febr 1981.

